DECAY ESTIMATES AND SMOOTHNESS FOR SOLUTIONS OF THE DISPERSION MANAGED NON-LINEAR SCHRÖDINGER EQUATION

DIRK HUNDERTMARK[1] AND YOUNG-RAN LEE[2]

ABSTRACT. We study the decay and smoothness of solutions of the dispersion managed non-linear Schrödinger equation in the case of zero residual dispersion. Using new x-space versions of bilinear Strichartz estimates, we show that the solutions are not only smooth, but also fast decaying.

1. INTRODUCTION

The parametrically excited one-dimensional non-linear Schrödinger equation (NLS) with periodically varying dispersion coefficient

$$iu_t + d(t)u_{xx} + c|u|^2 u = 0 \tag{1.1}$$

arises naturally as an envelope equation for electromagnetic wave propagation in optical waveguides used in fiber-optics communication systems where the dispersion is varied periodically along an optical fiber; it describes the amplitude of a signal transmitted via amplitude modulation of a carrier wave through a fiber-optical cable, see, e.g., [2, 34, 38]. In (1.1) t corresponds to the distance along the fiber, x denotes the (retarded) time, $u_t = \partial_t u = \frac{\partial}{\partial t}u$, $u_{xx} = \partial_x^2 u = \frac{\partial^2}{(\partial x)^2}u$, c a constant determining the strength of the non-linearity which, for convenience, we put equal to one in the following, and $d(t)$ the dispersion along the waveguide, which, for practical purposes, one can assume to be piecewise constant. The balance between dispersion and non-linearity is the key factor which determines the existence of stable soliton like pulses.

With fast data transfer through fiber-optic cables over long intercontinental distances in mind, one would like to use stable pulses, i.e., solitons, which do not change shape when traveling through the cable. The NLS does support solitons, but those depend on a delicate balance between dispersion and non-linearity. Also these solitary pulses then strongly interact with each other via the non-linear effects, which limits the bandwidth of the waveguide since each pulse must, therefore, be well separated from the next. Even worse, when multiplexing, that is, using multiple carrier waves with different frequencies, blue, green, and red, say, to create several channels in the

[1] supported in part by NSF grant DMS–0400940.
[2] supported in part by the Korean Science and Engineering Foundation(KOSEF) grant funded by the Korean government(MOST) (No.R01-2007-000-11307-0).

April 2008, version 15, file dms-15.tex.

waveguide which can be used simultaneously to increase the bit rate through the fiber, the pulses in each channel will travel with different group velocities determined by the carrier frequency and the dispersion relation in the optical cable. These pulses will overtake the ones on the 'slower' channel, hence pulses from different channels are bound to strongly interact.

One possibility to limit these negative effects of the non-linearity is to stay in the linear regime, with vanishing non-linearity, or at least the quasi-linear regime, where the non-linearity is small and hence the pulses do not interact, at least not much, with each other. On the other hand, there are no stable pulses in these regimes of small non-linearity where the dispersion dominates. All pulses broaden due to the dispersion, which again severely limits the bandwidth of long optical waveguides.

The technique of dispersion management was invented to overcome the difficulty that there are no stable pulses in the linear regime. The idea, building on the fact that optical fibers can be engineered to have positive and negative dispersion, see [7], is to use alternating sections of constant but opposite, or nearly opposite, dispersion. This introduces a rapidly varying dispersion $d(t)$ along the fiber, which, if the dispersion exactly cancel each other, leads to pulses changing periodically along the fiber. This idea had been introduced in 1980 in [21]. It has turned out to be enormously fruitful, see for example, [1, 10, 11, 17, 18, 19, 25, 27] and the references therein, even if one takes small non-linear effects into account and allows for a small residual dispersion along the fiber, the residual dispersion together with the non-linearity can balance each other, allowing the existence of stable soliton-like pulses. Record breaking transmission rates of more than 1 Tbits/s over an 18,000 kilometer optical fiber had been achieved using this technology [26] and the technique of dispersion management is now widely used commercially.

Due to the enormous practical implications, there has been a huge literature concerning the numerical and phenomenological explanations and the theoretical, but most often non-rigorous, understanding of the stabilizing effects of dispersion management techniques, mainly in the regime of strong dispersion management. In this regime neither the non-linearity nor the residual dispersion need to be small, but they are small relative to the local dispersion given by

$$d(t) = \frac{1}{\varepsilon} d_0(t) + d_{\mathrm{av}}. \tag{1.2}$$

Here $d_0(t)$ is the mean zero part, d_{av} the average dispersion over one period, and ε a usually small parameter. The envelope equation valid in this regime was derived by Gabitov and Turitsyn in 1996, [10, 11]. It is given by a non-linear Schrödinger equation which, after rescaling t to t/ε, takes the form

$$iu_t + d_0(t)u_{xx} + \varepsilon(d_{\mathrm{av}}u_{xx} + |u|^2u) = 0. \tag{1.3}$$

Note that the average dispersion and non-linearity is small compared to the local dispersion, which is a characteristic feature of the strong dispersion management regime.

Since the full equation (1.3) is very hard to study, one makes one approximation to the full equation: Assume that the mean zero part of dispersion along the fiber is -1 on the interval $[-1, 0]$ and $+1$ on $[0, 1]$. Then separating the free motion given by

the solution of $-iu_t + d_0(t)u_{xx} = 0$ and averaging (1.3) over the period, see [1, 10], yields

$$iv_t + \varepsilon d_{\mathrm{av}} v_{xx} + \varepsilon Q(v, v, v) = 0 \tag{1.4}$$

for the "averaged" solution v, where

$$Q(v_1, v_2, v_3) := \int_0^1 T_r^{-1}\big[T_r v_1 \overline{T_r v_2} T_r v_3\big]\, dr \tag{1.5}$$

and $T_r = e^{ir\partial_x^2}$ is the solution operator of the free Schrödinger equation and, by symmetry, we restrict the integration in the r-variable to $[0, 1]$. In some sense, v is a slowly varying variable along the optical waveguide and the varying dispersion, is interpreted, in the spirit of Kapitza's treatment of the unstable pendulum which is stabilized by fast small oscillations of the pivot, see [20], as a fast background oscillation, justifying formally the above averaging procedure.

The Gabitov–Turitzin model (1.4) for the dispersion managed optical waveguide is well-supported by numerical studies, see, for example, [1] and [37], and theoretical arguments, see, for example, [23, 24]. In addition, this averaging procedure was rigorously justified in [39] where it is shown that in the regime of strong dispersion management, $\varepsilon \ll 1$, on long scales $0 \le t \le C\varepsilon^{-1}$ the solution of the full equation (1.3) stays ε-close to a solution of (1.4) with the same initial data, showing that it is indeed an infinite dimensional analogue of Kapitza's effect. Moreover, the averaged equation (1.4) supports stable solitary solutions in certain regimes of the parameters. These solitary solutions give the average profile of the breather–like pulses in (1.1). Making the ansatz $v(t, x) = e^{i\varepsilon\omega t} f(x)$ in (1.4) yields the time independent equation

$$-\omega f = -d_{\mathrm{av}} f_{xx} - Q(f, f, f) \tag{1.6}$$

describing stationary soliton-like solutions, the so-called dispersion managed solitons. Equation (1.6) is the Euler-Lagrange equation for the averaged Hamiltonian

$$H(f) = \frac{d_{\mathrm{av}}}{2} \int_{\mathbb{R}} |f'|^2 dx - \frac{1}{4}\mathcal{Q}(f, f, f, f), \tag{1.7}$$

where we set

$$\mathcal{Q}(f_1, f_2, f_3, f_4) = \int_0^1 \int_{\mathbb{R}} \overline{T_r f_1(x)} T_r f_2(x) \overline{T_r f_3(x)} T_r f_4(x)\, dx dr. \tag{1.8}$$

Again T_r is the free Schrödinger evolution.

The very large literature of numerical and phenomenological explanations and the theoretical understanding of the stabilizing effects of dispersion management techniques in the strong dispersion management regime is mainly based on the averaged equation (1.6). Despite this enormous interest in dispersion managed solitons, there are few rigorous results available. Note that both $Q(f, f, f)$ and $\mathcal{Q}(f, f, f, f)$ are nonlinear and, in addition, highly non-local functions of f. This presents a unique challenge in the study of (1.6). Existence of solutions of (1.6) had first been rigorously established in [39] for positive residual dispersion $d_{\mathrm{av}} > 0$. Instead of showing existence of solutions of (1.6) directly, the existence of minimizers of the constraint

minimization problem

$$\overline{P}_{\lambda,d_{\mathrm{av}}} = \inf\Big\{H(f):\, f\in H^1(\mathbb{R}), \|f\|_2^2=\lambda\Big\} \tag{1.9}$$

was proved. Simple Gaussian testfunctions show $\overline{P}_{\lambda,d_{\mathrm{av}}}<0$. Since (1.6) is the Euler-Lagrange equation for constraint minimization problem (1.9), any minimizer of (1.9) is a weak solution of (1.6) with some $\omega>0$. Of course, minimizing sequences for the minimization problem (1.9) can very easily converge weakly to zero, since the functional (1.7) is invariant under shifts of f. This non-compactness was overcome using Lions' concentration compactness principle [22].

In the case of positive d_{av}, every weak solution $f\in H^1(\mathbb{R})$ of (1.6) with $\omega>0$ is automatically $\mathcal{C}^\infty$; recall that $\omega>0$ for any minimizer of (1.9). The smoothness follows from a simple bootstrapping argument. Using that $f\mapsto Q(f,f,f)$ maps the Sobolev spaces $H^s(\mathbb{R})$ into themselves and $(\omega-d_{\mathrm{av}}\partial_x^2)^{-1}$ maps $H^s(\mathbb{R})$ into $H^{s+2}(\mathbb{R})$, as long as $\omega>0$, straightforward bootstrapping shows that any solution $f\in H^1(\mathbb{R})$ of (1.6) with $\omega>0$ is in all the $H^s(\mathbb{R})$ Sobolev spaces for all $s\geq 1$, hence smooth by the Sobolev embedding theorem.

The variational problem in the case of vanishing residual dispersion, $d_{\mathrm{av}}=0$ is much more subtle and complicated due to an additional loss of compactness. Nevertheless, it is very important physically, since certain physical effects which destabilize pulse propagation in optical fibers are minimal for d_{av} near or equal to zero [31, 36]. In this case the constraint minimizing problem is given by

$$\overline{P}_{\lambda} = \inf\Big\{-\frac{1}{4}\mathcal{Q}(f,f,f,f):\, f\in L^2(\mathbb{R}), \|f\|_2^2=\lambda\Big\}. \tag{1.10}$$

Using the Strichartz inequality, it was shown in [39] that even in this case $\overline{P}_\lambda>-\infty$, see also Lemma 2.1 below. Now the minimizing sequence is only bounded in L^2 and, since the functional in (1.10) is invariant under shift of f in real space and in Fourier space, the traditional a-priori bounds from the calculus of variations are not available. The existence of a minimizer for the variational problem (1.10) was shown by Markus Kunze [16], using the concentration compactness principle in tandem; first in Fourier and then in x-space. This minimizer yields a solution for dispersion management equation (1.6) for vanishing average dispersion, $d_{\mathrm{av}}=0$. Unfortunately, the bootstrapping argument which shows smoothness of solutions of (1.6) for $d_{\mathrm{av}}>0$ now fails when $d_{\mathrm{av}}=0$ since there is a loss of the second order derivatives. The minimizer is now only in $L^2(\mathbb{R})$ and the nonlinearity Q is not smoothness improving, so Kunze's method does not give much more a-priori information on the minimizer besides being square integrable and bounded. Shortly afterwords, Milena Stanislavova showed that Kunze's minimizer is smooth. Her approach employed the use of Bourgain spaces [3, 4] and Tao's bilinear estimates [35].

To the best of our knowledge these results are the only known rigorous results concerning solutions of (1.6). For example, nothing is rigorously known so far on the decay properties of dispersion managed solitons. This is a tantalizing situation: since the tails of dispersion managed solitons are responsible for the interactions of pulses launched into the optical fiber, the tails essentially limit the bit rate capacity of optical waveguides. Thus finding the asymptotic behavior of dispersion managed

solitons is an important fundamental and practical problem which has attracted a lot of attention in numerical and phenomenological studies. Lushnikov [24] gave convincing but non-rigorous arguments that for any solution f of (1.6),

$$f(x) \sim A\cos(a_0x^2 + a_1x + a_2)\mathrm{e}^{-b|x|} \quad \text{as } x \to \infty \tag{1.11}$$

for some suitable choice of real constants a_j and $b > 0$, see also [23].

In this paper we derive *the first rigorous decay bounds* on dispersion managed solitons. Although our approach is, so far, not able to give exponential decay of dispersion managed solitons conjectured by Lushnikov, it shows that any solution of (1.6) in the case of vanishing residual dispersion, $d_{\mathrm{av}} = 0$, is super–polynomially decaying.

Theorem 1.1. *Let $\omega > 0$. Any weak solution $f \in L^2(\mathbb{R})$ of $\omega f = Q(f, f, f)$ is a Schwarz function. That is, f is arbitrary often differentiable and f and all its derivatives $f^{(n)}$ decay faster than polynomially at infinity,*

$$\sup_x |x|^m |f^{(n)}(x)| < \infty \qquad \textit{for all } m, n \in \mathbb{N}_0.$$

Remarks 1.2. (i) By a weak solution, we mean a function $f \in L^2(\mathbb{R})$ such that

$$\langle g, f\rangle = \langle g, Q(f, f, f)\rangle = \mathcal{Q}(g, f, f, f) \tag{1.12}$$

for all $g \in L^2(\mathbb{R})$. Here $\langle g, f\rangle = \int_{\mathbb{R}} \overline{g(x)} f(x)\, dx$ is the usual scalar product on $L^2(\mathbb{R})$. Note that our scalar product is sesquilinear in the first component and linear in the second. Also, due to the Strichartz inequality, the functional $\mathcal{Q}(f_1, f_2, f_3, f_4)$ is well-defined as soon as $f_j \in L^2(\mathbb{R})$ for all $j = 1, 2, 3, 4$, see Lemma 2.1. In turn, this means that $Q(f_1, f_2, f_3)$ is an L^2 function for all $f_1, f_2, f_3 \in L^2(\mathbb{R})$ and the notion of a weak solution of (1.6) by testing with L^2 functions, if $d_{\mathrm{av}} = 0$, respectively, $H^1(\mathbb{R})$ functions if $d_{\mathrm{av}} > 0$, makes sense.

(ii) Since $\mathcal{Q}(f, f, f, f) = 0$ implies $f \equiv 0$ by the unicity of T_t, any nontrivial weak solution of $\omega f = Q(f, f, f)$ automatically has $\omega = \omega_f = \mathcal{Q}(f, f, f, f)/\langle f, f\rangle > 0$.

(iii) One can give a more precise estimate on the super-polynomial decay rate of f, see Remark 1.4.ii and Corollary 3.3. This misses the conjectured exponential decay rate, however.

(iv) Theorem 1.1 significantly strengthens Stanislavova's result on smoothness of dispersion managed solitons in [32]. In addition, our proof is technically much simpler than Stanislavova's.

We will deduce the regularity property of dispersion managed solitons given in Theorem 1.1 from a suitable decay estimate on the tails of the solutions f and its Fourier transform $\widehat{f}$. For this we need some more notations. For $f \in L^2(\mathbb{R})$, let

$$\alpha(s) := \Big(\int_{|x|\ge s} |f(x)|^2\, dx\Big)^{1/2} \tag{1.13}$$

$$\beta(s) := \Big(\int_{|k|\ge s} |\widehat{f}(k)|^2\, dk\Big)^{1/2} \tag{1.14}$$

be the L^2-norm of its tail, respectively the tail of its Fourier transform $\widehat{f}$. For a general function $f \in L^2(\mathbb{R})$, the only thing one can say a-priori about α and β is that they both decay to zero as $s \to \infty$. In general, this decay can be arbitrarily slow. For weak solutions of the dispersion management equation more is true.

Proposition 1.3 (Super-algebraic decay of the tails). *Any weak solution $f \in L^2(\mathbb{R})$ of $\omega f = Q(f, f, f)$ obeys the a-priori estimates*

$$\alpha(s) \le C_\gamma s^{-\gamma} \tag{1.15}$$

$$\beta(s) \le C_\gamma s^{-\gamma} \tag{1.16}$$

for all $s > 0$, all $\gamma > 0$, and some finite constant C_γ.

Remarks 1.4. (i) In fact, a slightly stronger result holds, see Corollary 3.3.
(ii) We get a decay estimate for f similar to the one for α, since

$$|f(s)|^2 + |f(-s)|^2 = 2\int_{|x|>s} \mathrm{Re} f'(x) f(x)\, dx \le 2\int_{|x|>s} |f'(x)f(x)|\, dx \le 2\|f'\|_2^2 \alpha(s).$$

An immediate corollary of Proposition 1.3 is that for any weak solution of $\omega f = Q(f, f, f)$ both f and $\hat{f}$ are in the Sobolev spaces $H^s(\mathbb{R})$ for arbitrary $s \ge 0$, in particular, both f and $\hat{f}$ are infinitely often differentiable.

Proposition 1.5. *Any weak solution $f \in L^2(\mathbb{R})$ of $\omega f = Q(f, f, f)$ is in the Sobolev space $H^s(\mathbb{R})$ for any $s \ge 0$. The same holds for $\hat{f}$. In particular, f and $\hat{f}$ are in $\mathcal{C}^\infty(\mathbb{R})$.*

In turn, Theorem 1.1 is a direct consequence of Proposition 1.5, see Lemma 3.7. In the next section we establish our main technical tools, multi-linear refinements of Strichartz estimates both in Fourier space and in x-space, Corollary 2.8, and the quasi-locality of the non-local functional $\mathcal{Q}$, Lemma 2.10. In Section 3 we use the above results to prove self-consistency bounds on the tail-distributions, Lemma 3.1. These self-consistency bounds are similar in spirit to sub-harmonicity bounds and are the main tool in our proof of the super-algebraic decay of dispersion managed solitons given in Corollary 3.3, which is a refinement of Proposition 1.3.

2. Multi-linear estimates

We want to study the smoothness and decay properties of general solutions of the (averaged) dispersion management equation (1.6) in the case of vanishing average dispersion, $d_{\mathrm{av}} = 0$. That is, we assume that $f \in L^2(\mathbb{R})$ is a weak solution of

$$\omega f = Q(f, f, f) \tag{2.1}$$

with $Q(f, f, f)$ given by (1.5). As mentioned in Remark 1.2.i, the right hand side of (2.1) is an $L^2(\mathbb{R})$ function for any $f \in L^2(\mathbb{R})$, thus the notion of a weak solution of (2.1) makes sense; $f \in L^2(\mathbb{R})$ is a weak solution of (2.1) if

$$\omega\langle g, f\rangle = \langle g, Q(f, f, f)\rangle = \mathcal{Q}(g, f, f, f) \tag{2.2}$$

for all $g \in L^2(\mathbb{R})$. The second equality in (2.2) follows from the unicity of $T_t = e^{it\partial_x^2}$. Since $\mathcal{Q}(f, f, f, f) > 0$ for $f \not\equiv 0$, $\omega > 0$.

The ground-state dispersion managed soliton is a solution of the minimization problem (1.10) or, equivalently, the maximization problem

$$P_\lambda = \sup\Big\{\mathcal{Q}(f,f,f,f) : f \in L^2(\mathbb{R}), \|f\|_2^2 = \lambda\Big\}. \tag{2.3}$$

By scaling, $\widetilde{f} = f/\sqrt{\lambda}$, one sees $P_\lambda = P_1\lambda^2$. Thus if f is a ground-state dispersion managed solitons then, testing (2.1) with f, one sees that f solves (2.1) with ω given by

$$\omega = P_1\lambda = P_1\|f\|_2^2. \tag{2.4}$$

We give several preparatory lemmas in this section.

Lemma 2.1. *The two-sided bound* $1.05(2\pi)^{-1/2} \le P_1 \le 12^{-1/4}$ *holds. In particular, for any functions* $f_j \in L^2(\mathbb{R})$, *j=1,2,3,4,*

$$|\mathcal{Q}(f_1,f_2,f_3,f_4)| \le P_1\prod_{j=1}^4 \|f_j\| \le 12^{-1/4}\prod_{j=1}^4\|f_j\|. \tag{2.5}$$

Proof. Using the triangle and generalized Hölder inequalities,

$$\begin{aligned}|\mathcal{Q}(f_1,f_2,f_3,f_4)| &\le \int_0^1\int_{\mathbb{R}}\prod_{j=1}^4 |T_tf_j|\,dxdt\\ &\le \prod_{j=1}^4\left(\int_0^1\int_{\mathbb{R}}|T_tf_j|^4dxdt\right)^{1/4} = \prod_{j=1}^4\left(\mathcal{Q}(f_j,f_j,f_j,f_j)\right)^{1/4}.\end{aligned}$$

Given f let $\widetilde{f} = f/\|f\|_2$, by definition of P_1,

$$\mathcal{Q}(f,f,f,f) = \mathcal{Q}(\widetilde{f},\widetilde{f},\widetilde{f},\widetilde{f})\|f\|_2^4 \le P_1\|f\|_2^4$$

which gives the first inequality in (2.5). The second inequality in (2.5) follows once the upper bound on P_1 is proven. For this we use the one-dimensional Strichartz inequality,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|T_tf(x)|^6\,dxdt \le S_1^6\|f\|_{L^2(\mathbb{R})}^6, \tag{2.6}$$

which holds due to the dispersive properties of the free Schrödinger equation, [12, 33, 34]. The sharp constant in (2.6) is known, $S_1 = 12^{-1/12}$, one even knows S_2 in two space dimensions, see [9, 14], but, so far, not in any other space dimension $d \ge 3$. Let $\|f\|_2 = 1$. Using the Cauchy-Schwarz inequality, one gets

$$\mathcal{Q}(f,f,f,f) = \int_0^1\int_{\mathbb{R}}|T_tf|^{3+1}dxdt \le \big(\int_0^1\int_{\mathbb{R}}|T_tf|^6dxdt\big)^{1/2}\big(\int_0^1\int_{\mathbb{R}}|T_tf|^2dxdt\big)^{1/2}.$$

The first factor is bounded with the help of (2.6) by extending the integral in t to all of $\mathbb{R}$. The second factor is bounded by doing the x-integration first, using that T_t is a unitary operator on $L^2(\mathbb{R})$. Thus

$$\mathcal{Q}(f,f,f,f) \le S_1^3 \quad \text{for all } \|f\|_2 = 1.$$

Hence $P_1 \le S_1^3 = 12^{-1/4}$, using the sharp value for the Strichartz constant. This proves the upper bound on P_1 and thus the second inequality in (2.5).

For the lower bound on P_1, we use a chirped Gaussian test-function similar to [16, 39]. If the initial condition f is given by

$$f(x) = A_0 e^{-x^2/\sigma_0} \quad \text{with } \mathrm{Re}(\sigma_0) > 0 \tag{2.7}$$

then with $\sigma(t) = \sigma_0 + 4it$ and $A(t) = A_0\sqrt{\sigma_0}/\sqrt{\sigma(t)}$, its free time-evolution is given by

$$T_t f(x) = A(t) e^{-x^2/\sigma(t)}, \tag{2.8}$$

see, e.g., [39]. Thus

$$\int_0^1 \int_{\mathbb{R}} |T_t f|^4 \, dxdt = \sqrt{\frac{\pi}{4}} |A_0|^4 |\sigma_0|^2 \int_0^1 \frac{1}{|\sigma(t)|} \, dt. \tag{2.9}$$

Choosing $|A_0|^2 = \sqrt{2\mathrm{Re}(\sigma_0)/(|\sigma_0|^2)\pi}$ yields the normalization $\|f\|_2 = 1$ and hence

$$P_1 \geq \frac{\mathrm{Re}(\sigma_0)}{\sqrt{\pi}} \int_0^1 \frac{1}{\sqrt{\mathrm{Re}(\sigma_0)^2 + (\mathrm{Im}(\sigma_0) + 4t)^2}} \, dt. \tag{2.10}$$

The best choice for $\mathrm{Im}(\sigma_0)$ is $\mathrm{Im}(\sigma_0) = -2$ and with $\delta = 2/\mathrm{Re}(\sigma_0)$ we arrive at

$$P_1 \geq \frac{1}{\sqrt{2\pi}} \sup_{\delta > 0} \frac{1}{\sqrt{\delta}} \int_0^{\delta} \frac{1}{\sqrt{1+s^2}} \, ds > \frac{1.05}{\sqrt{2\pi}}, \tag{2.11}$$

which, noticing that the supremum is attained at approximately $\delta = 3.32$, gives the claimed lower bound on P_1. ∎

Besides some estimates on $\mathcal{Q}$, we also need, for technical reasons, bounds on the slightly modified functional

$$\mathcal{R}(f_1, f_2, f_3, f_4) := \int_0^1 \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) \, t dxdt, \tag{2.12}$$

where the measure $dxdt$ on $\mathbb{R} \times [0,1]$ is changed to $tdxdt$.

Lemma 2.2. *For any functions $f_j \in L^2(\mathbb{R})$, j=1,2,3,4,*

$$|\mathcal{R}(f_1, f_2, f_3, f_4)| \leq \frac{12^{-1/4}}{\sqrt{3}} \prod_{j=1}^4 \|f_j\|. \tag{2.13}$$

Proof. Again, as in the proof of Lemma 2.1, using the triangle and generalized Hölder inequalities, one sees

$$|\mathcal{R}(f_1, f_2, f_3, f_4)| \leq \prod_{j=1}^4 \mathcal{R}\left(f_j, f_j, f_j, f_j\right)^{1/4}.$$

So it is enough to prove (2.13) in the case $f_j = f$ for all $j = 1,2,3,4$. Using the Cauchy-Schwarz inequality,

$$\mathcal{R}(f,f,f,f) = \int_0^1 \int_{\mathbb{R}} |T_t f|^{3+1} t \, dxdt \leq \Big(\int_0^1 \int_{\mathbb{R}} |T_t f|^6 \, dxdt \Big)^{1/2} \Big(\int_0^1 \int_{\mathbb{R}} |T_t f|^2 t^2 \, dxdt \Big)^{1/2}.$$

Again, the first factor is bounded by extending the t-integration to all of $\mathbb{R}$ and then using the one-dimensional Strichartz inequality (2.6) and the second doing the x-integration first, using the unicity of T_t. Thus

$$\mathcal{R}(f,f,f,f) \le \frac{S_1^3}{\sqrt{3}} \|f\|^4_{L^2(\mathbb{R})}.$$

Since $S_1^3 = 12^{-1/4}$, this proves (2.13). ∎

The following estimates are our main tools to prove the regularity properties of dispersion managed solitons. The results below have natural generalizations to arbitrary dimension. We need only their one-dimensional versions. Recall that $T_t = e^{it\partial_x^2}$ is the solution operator for the free Schrödinger equation in dimension one, that is

$$T_t f(x) = \frac{1}{\sqrt{4\pi i t}} \int_{\mathbb{R}} e^{i\frac{|x-y|^2}{4t}} f(y)\, dy \tag{2.14}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\eta} e^{-it\eta^2} \widehat{f}(\eta)\, d\eta. \tag{2.15}$$

Here $\widehat{f}$ is the Fourier transform of f, given by

$$\widehat{f}(\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\eta} f(x)\, dx, \tag{2.16}$$

for $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and extended to a unitary operator to all of $L^2(\mathbb{R})$. The inverse Fourier transform is given by $\check{f}$,

$$\check{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\eta} f(\eta) d\eta. \tag{2.17}$$

The following bilinear estimate for initial conditions f_1 and f_2 whose Fourier transforms have separated supports is well known.

Lemma 2.3 (Fourier space bilinear estimate)**.** *If the initial conditions $f_1, f_2 \in L^2(\mathbb{R})$ have separated supports in Fourier space,* $\operatorname{dist}(\operatorname{supp} \widehat{f_1}, \operatorname{supp} \widehat{f_2}) > 0$*, then*

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}, dtdx)} \le \frac{1}{\sqrt{2 \operatorname{dist}(\operatorname{supp} \widehat{f_1}, \operatorname{supp} \widehat{f_2})}} \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}. \tag{2.18}$$

The above bound is one of the key ingredients to prove the Fourier-space part of Theorem 1.1. For the x-space bounds, we need an x-space version of the above bilinear estimate. For this the following observation is helpful.

Lemma 2.4 (Duality)**.** *Let $f_1, f_2 \in L^2(\mathbb{R})$ with $\check{f}_1$ and $\check{f}_2$ the corresponding inverse Fourier transforms. Then*

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}, |t|^{-1} dtdx)} = \sqrt{2} \|T_t \check{f}_1 T_t \check{f}_2\|_{L^2(\mathbb{R}\times\mathbb{R}, dtdx)}. \tag{2.19}$$

Remark 2.5. Note that in the L^2-norm on the left hand side, the measure $dtdx$ is replaced by the measure $|t|^{-1} dtdx$, which is highly singular at $t = 0$.

Together with Lemma 2.3, this duality result gives a real space version of the bilinear estimates.

Lemma 2.6 (x-space bilinear estimate)**.** *If the initial conditions $f_1, f_2 \in L^2(\mathbb{R})$ have separated supports,* $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) > 0$*, then*

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R},|t|^{-1}dtdx)} \le \frac{1}{\sqrt{\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2)}} \|f_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})}. \tag{2.20}$$

Proof of Lemma 2.6. Assuming Lemma 2.3 and 2.4 for the moment, the proof is straightforward. Since the Fourier transform of $\check{f}_j$ is f_j for $j = 1, 2$, the assumption that f_1 and f_2 have well separated supports, $\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2) > 0$, means that the supports of the Fourier transforms of $\check{f}_1$ and $\check{f}_2$ are well separated. Thus Lemma 2.3 applies to the right hand side of (2.19) and hence

$$\begin{aligned}\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R},|t|^{-1}dtdx)} &= \sqrt{2}\|T_t \check{f}_1 T_t \check{f}_2\|_{L^2(\mathbb{R}\times\mathbb{R},dtdx)} \\ &\le \frac{1}{\sqrt{\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2)}} \|f_1\|_{L^2(\mathbb{R})}\|f_2\|_{L^2(\mathbb{R})}.\end{aligned}$$

∎

It remains to prove the duality Lemma and the bilinear estimate in Fourier space.

Proof of Lemma 2.4. Using the explicit form of the free time evolution (2.14), we see that

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} |T_t f_1 T_t f_2|^2\, |t|^{-1} dxdt \\ &= \frac{1}{(4\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|\int_{\mathbb{R}^2} e^{ix(y_1+y_2)/(2t)} e^{-i(y_1^2+y_2^2)/(4t)} f_1(y_1) f_2(y_2)\, dy_1 dy_2\Big|^2 \frac{dxdt}{|t|^3} \\ &= 2\int_{\mathbb{R}}\int_{\mathbb{R}} \Big|\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{iz(y_1+y_2)} e^{-i\tau(y_1^2+y_2^2)} f_1(y_1) f_2(y_2)\, dy_1 dy_2\Big|^2 dzd\tau \end{aligned} \tag{2.21}$$

where we first made the change of variables $x = 2tz$, $dx = 2|t|dz$, and then $t = 1/(4\tau)$ with $t^{-2}dt = 4d\tau$. Let $\check{f}_j$ be the inverse Fourier transform of f. Since

$$T_\tau \check{f}_j(z) = \frac{1}{(2\pi)^{1/2}}\int_{\mathbb{R}} e^{izy} e^{-i\tau y^2} f_j(y)\, dy$$

one has

$$T_\tau \check{f}_1(z) T_\tau \check{f}_2(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{iz(y_1+y_2)} e^{-i\tau(y_1^2+y_2^2)} f_1(y_1) f_2(y_2)\, dy_1 dy_2$$

and plugging this back into (2.21) gives

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |T_t f_1 T_t f_2|^2\, |t|^{-1} dxdt = 2\int_{\mathbb{R}}\int_{\mathbb{R}} |T_\tau \check{f}_1(z) T_\tau \check{f}_2(z)|^2\, dzd\tau$$

which is 2.19. ∎

Proof of Lemma 2.3. This result is known to the experts, see, for example [8, 28]. We give a proof for the convenience of the reader. Using the Fourier representation (2.15) of a solution of the free Schrödinger equation,

$$T_t f_1(x) T_t f_2(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{ix(k_1+k_2)} e^{-it(k_1^2+k_2^2)} \widehat{f}_1(k_1)\widehat{f}_2(k_2)\, dk_1 dk_2.$$

In particular,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |T_t f_1 T_t f_2|^2 \, dxdt = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}}\Big|\int_{\mathbb{R}^2} e^{ix(k_1+k_2)} e^{-it(k_1^2+k_2^2)} \widehat{f}_1(k_1)\widehat{f}_2(k_2)\, dk_1 dk_2\Big|^2 \, dxdt.$$

Expanding the square, using $\delta(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{isk}\, ds$ as distributions, this leads to

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |T_t f_1 T_t f_2|^2 \, dxdt = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \delta(\eta_1+\eta_2-\zeta_1-\zeta_2)\delta(\eta_1^2+\eta_2^2-\zeta_1^2-\zeta_2^2)\overline{\widehat{f}_1(\eta_1)\widehat{f}_2(\eta_2)}\widehat{f}_1(\zeta_1)\widehat{f}_2(\zeta_2)\, d\eta_1 d\eta_2 d\zeta_1 d\zeta_2. \tag{2.22}$$

Now we make the change of variables $\xi_1 = \eta_1 + \eta_2$, $\vartheta_1 = \eta_1^2 + \eta_2^2$, and $\xi_2 = \zeta_1 + \zeta_2$, $\vartheta_2 = \zeta_1^2 + \zeta_2^2$. By the inverse function theorem, the inverse of the Jacobian of the change of variables $(\xi_1, \vartheta_1) \mapsto (\eta_1, \eta_2)$ is given by $J^{-1} = \begin{pmatrix} \partial\xi_1/\partial\eta_1 & \partial\xi_1/\partial\eta_2 \\ \partial\vartheta_1/\partial\eta_1 & \partial\vartheta_1/\partial\eta_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2\eta_1 & 2\eta_2 \end{pmatrix}$. That is, $\det J^{-1} = 2(\eta_2-\eta_1)$ and hence

$$d\eta_1 d\eta_2 = |\det J|\, d\xi_1 d\vartheta_1 = \frac{d\xi_1 d\vartheta_1}{2|\eta_2-\eta_1|}.$$

Thus, setting $\widehat{f}_1 \otimes \widehat{f}_2(\xi_1,\vartheta_1) = \widehat{f}_1(\eta_1(\xi_1,\vartheta_1))\widehat{f}_2(\eta_2(\xi_1,\vartheta_1))$ and similarly for $\widehat{f}_1 \otimes \widehat{f}_2(\xi_2,\vartheta_2)$, we can rewrite (2.22) as

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \delta(\eta_1+\eta_2-\zeta_1-\zeta_2)\delta(\eta_1^2+\eta_2^2-\zeta_1^2-\zeta_2^2)\overline{\widehat{f}_1(\eta_1)\widehat{f}_2(\eta_2)}\widehat{f}_1(\zeta_1)\widehat{f}_2(\zeta_2)\, d\eta_1 d\eta_2 d\zeta_1 d\zeta_2 \\
&= \int_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}\times\mathbb{R}_+} \delta(\xi_1-\xi_2)\delta(\vartheta_1-\vartheta_2)\overline{\widehat{f}_1\otimes\widehat{f}_2}(\xi_1,\vartheta_1)\widehat{f}_1\otimes\widehat{f}_2(\xi_2,\vartheta_2)\,\frac{d\xi_1 d\vartheta_1 d\xi_2 d\vartheta_2}{4|\eta_2-\eta_1||\zeta_2-\zeta_1|} \\
&= \int_{\mathbb{R}\times\mathbb{R}_+} |\widehat{f}_1\otimes\widehat{f}_2(\xi_1,\vartheta_1)|^2\,\frac{d\xi_1 d\vartheta_1}{4|\eta_2-\eta_1|^2} \\
&\le \frac{1}{2\,\mathrm{dist}(\mathrm{supp}\,\widehat{f}_1, \mathrm{supp}\,\widehat{f}_2)}\int_{\mathbb{R}\times\mathbb{R}_+} |\widehat{f}_1\otimes\widehat{f}_2(\xi_1,\vartheta_1)|^2\,\frac{d\xi_1 d\vartheta_1}{2|\eta_2-\eta_1|} \\
&= \frac{1}{2\,\mathrm{dist}(\mathrm{supp}\,\widehat{f}_1, \mathrm{supp}\,\widehat{f}_2)}\|\widehat{f}_1\|_2^2\|\widehat{f}_2\|_2^2
\end{aligned}$$

where, in the last equality, we undid the change of variables $\xi_1 = \eta_1+\eta_2$, $\vartheta_1 = \eta_1^2+\eta_2^2$. This proves (2.18). ∎

Remarks 2.7. (i) The Fourier space version of the bilinear estimate has a generalization to arbitrary space dimension. If $f_j \in L^2(\mathbb{R}^d)$ are well separated in Fourier space and $T_t = e^{it\Delta}$, with $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ the Laplacian in $\mathbb{R}^d$, the free Schrödinger time evolution, then

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, dtdx)} \le \frac{C}{\sqrt{\mathrm{dist}(\mathrm{supp}\,\widehat{f}_1, \mathrm{supp}\,\widehat{f}_2)}}\|f_1\|_{L^2(\mathbb{R}^d)}\|f_2\|_{L^2(\mathbb{R}^d)}. \tag{2.23}$$

for some constant depending on d, see, for example, [15, 8].

(ii) Similarly, a suitable version of the duality Lemma is valid in all space dimensions. To formulate this, let $\widehat{f}$ be the d-dimensional Fourier transform of f, given by

$$\widehat{f}(\eta) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\eta} f(x)\, dx$$

for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and extended to a unitary operator to all of $L^2(\mathbb{R}^d)$. The inverse Fourier transform is again denoted by $\check{f}$,

$$\check{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\eta} f(\eta) d\eta \ .$$

Then

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, |t|^{d-2} dtdx)} = 2^{1-d/2} \|T_t \check{f}_1 T_t \check{f}_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, dtdx)}. \tag{2.24}$$

This follows from a similar calculation as in the proof of Lemma 2.4 using now the representations

$$T_t f(x) = \frac{1}{(4\pi i t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{4t}} f(y)\, dy = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\eta} e^{-it\eta^2} \widehat{f}(\eta)\, d\eta \tag{2.25}$$

for the free Schrödinger evolution in $\mathbb{R}^d$.

(iii) The bilinear estimate for initial conditions which are separated in x-space is our main tool to get decay estimates on the dispersion managed soliton in x-space. By the two remarks above, the proof of Lemma 2.6 immediately generalizes to all space dimensions giving the following bilinear real space estimate: If $f_j \in L^2(\mathbb{R}^d)$ have separated supports and $T_t = e^{it\Delta}$, with $\Delta = \sum_{j=1}^d \partial_{x_j}^2$ the Laplacian in $\mathbb{R}^d$, the free Schrödinger time evolution, then

$$\|T_t f_1 T_t f_2\|_{L^2(\mathbb{R}\times\mathbb{R}^d, |t|^{d-2} dtdx)} \le \frac{C}{\sqrt{\operatorname{dist}(\operatorname{supp} f_1, \operatorname{supp} f_2)}} \|f_1\|_{L^2(\mathbb{R}^d)} \|f_2\|_{L^2(\mathbb{R}^d)} \tag{2.26}$$

for some constant C.

(iv) The duality under Fourier transform in Lemma 2.4 was first noticed in the context of the Strichartz estimate in [14] for the Strichartz norm in dimension one and two. In fact, it holds in general for suitable mixed space-time norms

$$\|u\|_{L_t^r L_x^p} = \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |u(t,x)|^p\, dx \Big)^{r/p} dt \Big)^{1/r}. \tag{2.27}$$

If $u(t,x) = T_t f(x)$ is the solution of the free Schrödinger equation in $\mathbb{R}^d$ then, using (2.25), a similar change of variables calculation as in the proof of Lemma 2.4 yields the symmetry

$$\|T_t f\|_{L_t^r L_x^p} = \|T_t \check{f}\|_{L_t^r L_x^p} \quad \text{ for } \frac{2}{r} = \frac{d}{2} - \frac{d}{p} \ . \tag{2.28}$$

The observation made here, that this type of invariance immediately transforms Fourier-space bilinear estimates into corresponding x-space bilinear bounds seems to be new.

(v) In a forthcoming paper, [13], we use the Fourier and x-space bilinear Strichartz

estimates to give a simple proof of existence of minimizers of the minimization problem (1.10) which avoids the use of Lion's concentration compactness principle.

For the application we have in mind, we need to have similar estimates for the functional $\mathcal{Q}$.

Corollary 2.8 (Multi-linear estimates)**.** *Let* $f_j \in L^2(\mathbb{R})$ *for* $j = 1, 2, 3, 4$.
(i) *If there exists a pair* $i \neq j$ *such that* $\operatorname{dist}(\operatorname{supp} f_i, \operatorname{supp} f_j) > 0$*, then*

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq \frac{1}{2^{1/4}\, 3^{3/8} \sqrt{\operatorname{dist}(\operatorname{supp} f_i, \operatorname{supp} f_j)}} \|f_1\|\|f_2\|\|f_3\|\|f_4\|. \tag{2.29}$$

(ii) *If there exists a pair* $i \neq j$ *such that* $\operatorname{dist}(\operatorname{supp} \widehat{f}_i, \operatorname{supp} \widehat{f}_j) > 0$*, then*

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq \frac{1}{2^{3/4} 3^{1/8} \sqrt{\operatorname{dist}(\operatorname{supp} \widehat{f}_i, \operatorname{supp} \widehat{f}_j)}} \|f_1\|\|f_2\|\|f_3\|\|f_4\|. \tag{2.30}$$

Proof. Since $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq \int_0^1 \int_{\mathbb{R}} |T_t f_1 T_t f_2 T_t f_3 T_t f_4| \, dt dx$ we can, without loss of generality, assume $i = 1$ and $j = 2$. First we prove (2.29). Using the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathcal{Q}(f_1, f_2, f_3, f_4) &\leq \int_0^1 \int_{\mathbb{R}} |T_t f_1 T_t f_2 T_t f_3 T_t f_4| \, dx dt \\ &\leq \Big(\int_0^1 \int_{\mathbb{R}} \frac{|T_t f_1 T_t f_2|^2}{t} \, dx dt \Big)^{1/2} \Big(\int_0^1 \int_{\mathbb{R}} t |T_t f_3 T_t f_4|^2 \, dx dt \Big)^{1/2}. \end{aligned} \tag{2.31}$$

The first factor is bounded by (2.20). The second factor equals $(\mathcal{R}(f_3, f_3, f_4, f_4))^{1/2}$, which is bounded by (2.13). This shows (2.29). The proof of (2.30) is analogous, using (2.18) and (2.5). ∎

Remark 2.9. We always have the bound $|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq P_1 \|f_1\|\|f_2\|\|f_3\|\|f_4\|$ by Lemma 2.1. So the bounds (2.29) and (2.30) can be improved for small separation of the supports. Chasing the constants, one sees

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq P_1 \min\big(1, \frac{1.33}{\sqrt{\operatorname{dist}(\operatorname{supp} f_i, \operatorname{supp} f_j)}}\big) \|f_1\|\|f_2\|\|f_3\|\|f_4\| \tag{2.32}$$

and

$$|\mathcal{Q}(f_1, f_2, f_3, f_4)| \leq P_1 \min\big(1, \frac{1.1}{\sqrt{\operatorname{dist}(\operatorname{supp} \widehat{f}_i, \operatorname{supp} \widehat{f}_j)}}\big) \|f_1\|\|f_2\|\|f_3\|\|f_4\| \tag{2.33}$$

but for our purposes precise estimates for the constants are not needed since they only indirectly affect the bound on the decay rate, see the proof of Corollary 3.3.

The next result is the second main ingredient for our bounds on dispersion managed solitons. It shows that although the functional $\mathcal{Q}(f_1, f_2, f_3, f_4)$ is highly non-local, it retains at least some locality both in Fourier and x-space.

Lemma 2.10 (Quasi-locality of $\mathcal{Q}$)**.** *Let* $s>0$ *and* $i=1,2,3$ *or* 4.

(i) *If* $\operatorname{supp} f_i \subset \{|x|>3s\}$ *and* $\operatorname{supp} f_j \subset \{|x|\le s\}$ *for all* $j\neq i$*, then*
$$\mathcal{Q}(f_1,f_2,f_3,f_4)=0.$$

(ii) *If* $\operatorname{supp} \hat{f}_i \subset \{|k|>3s\}$ *and* $\operatorname{supp} \hat{f}_j \subset \{|k|\le s\}$ *for all* $j\neq i$*, then*
$$\mathcal{Q}(f_1,f_2,f_3,f_4)=0.$$

Proof. We give the proof for $i=1$, the other cases are similar. For part (i) of the lemma, we express $\mathcal{Q}(f_1,f_2,f_3,f_4)$ using (2.14) similar to the proof of the Duality Lemma 2.4.

$$\begin{aligned}
&\mathcal{Q}(f_1,f_2,f_3,f_4)\\
&=\int_0^1 \frac{dt}{(4\pi t)^2}\int_{\mathbb{R}} dx \int_{\mathbb{R}^4} e^{\frac{ix(y_1-y_2+y_3-y_4)}{2t}} e^{\frac{-i(y_1^2-y_2^2+y_3^2-y_4^2)}{4t}}\overline{f_1(y_1)}f_2(y_2)\overline{f_3(y_3)}f_4(y_4)dy\\
&=\frac{1}{8\pi^2}\int_0^1 \frac{dt}{t}\int_{\mathbb{R}} dz \int_{\mathbb{R}^4} e^{i(y_1-y_2+y_3-y_4)z} e^{\frac{-i(y_1^2-y_2^2+y_3^2-y_4^2)}{4t}}\overline{f_1(y_1)}f_2(y_2)\overline{f_3(y_3)}f_4(y_4)dy\\
&=\frac{1}{4\pi}\int_0^1 \frac{dt}{t}\int_{\mathbb{R}^4} \delta(y_1-y_2+y_3-y_4) e^{\frac{-i(y_1^2-y_2^2+y_3^2-y_4^2)}{4t}}\overline{f_1(y_1)}f_2(y_2)\overline{f_3(y_3)}f_4(y_4)dy
\end{aligned} \tag{2.34}$$

where we made the change of variables with $x=2tz$. The δ-functions restrict the integration to the subspace $y_1=y_2-y_3+y_4$. Because of our assumption on the supports of f_j, the product of the f_j, and hence the integrand, vanishes for any (y_1,y_2,y_3,y_4) with $y_1=y_2-y_3+y_4$. This proves part (i).

Analogously, for part (ii), we use the representation (2.14) to see

$$\begin{aligned}
&\mathcal{Q}(f_1,f_2,f_3,f_4)\\
&=\frac{1}{(2\pi)^2}\int_0^1 dt\int_{\mathbb{R}} dx \int_{\mathbb{R}^4} e^{-ix(\eta_1-\eta_2+\eta_3-\eta_4)} e^{it(\eta_1^2-\eta_2^2+\eta_3^2-\eta_4^2)}\overline{\widehat{f}_1(\eta_1)}\widehat{f}_2(\eta_2)\overline{\widehat{f}_3(\eta_3)}\widehat{f}_4(\eta_4)d\eta\\
&=\frac{1}{2\pi}\int_0^1 dt\int_{\mathbb{R}^4} \delta(\eta_1-\eta_2+\eta_3-\eta_4) e^{it(\eta_1^2-\eta_2^2-\eta_3^2-\eta_4^2)}\overline{\widehat{f}_1(\eta_1)}\widehat{f}_2(\eta_2)\overline{\widehat{f}_3(\eta_3)}\widehat{f}_4(\eta_4)d\eta\\
&=0
\end{aligned}$$

under the condition of the support of $\widehat{f}_j$, $j=1,2,3,4$.

■

Remarks 2.11. (i) As the above proof shows, $\mathcal{Q}(f_1,f_2,f_3,f_4)=0$ if either $0\notin \operatorname{supp}(f_1)-\operatorname{supp}(f_2)+\operatorname{supp}(f_3)-\operatorname{supp}(f_4)$ or $0\notin \operatorname{supp}(\widehat{f}_1)-\operatorname{supp}(\widehat{f}_2)+\operatorname{supp}(\widehat{f}_3)-\operatorname{supp}(\widehat{f}_4)$.

(ii) That the functional $\mathcal{Q}$ is quasi-local in Fourier space is not necessarily a surprise. In Fourier space the space integral of the product of the time evolved wave packets T_tf_j amounts to a convolution of the respective Fourier transforms. The additional δ-function in the variables $\eta_1-\eta_2+\eta_3-\eta_4$ expressed momentum conservation, since $\mathcal{Q}$ is invariant under translations. That the same result holds for wave packets corresponding to initial conditions which are separated in real space is more surprising,

since the free Schrödinger equation is dispersive and the wave packets $T_t f_j$ have a lot of overlap for $t \neq 0$, even if they are well separated for $t = 0$.

(iii) There is a related duality result for $\mathcal{Q}$ similar to the duality Lemma 2.4 for the bilinear norms, which explains a bit the quasi-locality of $\mathcal{Q}$ in real space. For this it is natural to consider a more general class of functionals given by

$$\mathcal{Q}_\psi(f_1, f_2, f_3, f_4) = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{T_t f_1(x)} T_t f_2(x) \overline{T_t f_3(x)} T_t f_4(x) \, \psi(t) dx dt$$

for a suitable cuff-off function ψ. Similar to the proof of Lemma 2.1, it is easy to see that $\mathcal{Q}$ is bounded on $L^2(\mathbb{R})$. Expressing $T_t f_j$ via (2.14) one has

$$\begin{aligned}
&\mathcal{Q}_\psi(f_1, f_2, f_3, f_4) \\
&\quad = \frac{1}{(4\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^4} e^{ix(y_1 - y_2 + y_3 - y_4)/(2t)} e^{-i(y_1^2 - y_2^2 + y_3^2 - y_4^2)/(4t)} \\
&\qquad\qquad \overline{f_1(y_1)} f_2(y_2) \overline{f_3(y_3)} f_4(y_4) \, dy \, \frac{\psi(t)}{|t|^2} dx dt \\
&\quad = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}^4} e^{iz(y_1 - y_2 + y_3 - y_4)} e^{-i\tau(y_1^2 - y_2^2 + y_3^2 - y_4^2)} \\
&\qquad\qquad \overline{f_1(y_1)} f_2(y_2) \overline{f_3(y_3)} f_4(y_4) \, dy \, \frac{\psi(1/(4\tau))}{2|\tau|} dz d\tau
\end{aligned}$$

where we first changed variables $x = 2tz$, $dx = 2|t|dz$ and then $\tau = 1/(4t)$, $\frac{d\tau}{|\tau|} = \frac{dt}{|t|}$. Hence with $\widetilde{\psi}(\tau) = \psi(1/(4\tau))/(2|\tau|)$ and recalling (2.15),

$$\mathcal{Q}_\psi(f_1, f_2, f_3, f_4) = \mathcal{Q}_{\widetilde{\psi}}(\check{f}_1, \check{f}_2, \check{f}_3, \check{f}_4) \tag{2.35}$$

where $\check{f}_j$ is the inverse Fourier transform of f_j. In particular, any result on $\mathcal{Q}_{\widetilde{\psi}}$ under conditions on the Fourier transforms of the involved functions implies the same result for $\mathcal{Q}_\psi$ under *exactly* the same conditions on the original functions f_j. For example, quasi-locality of $\mathcal{Q}_{\widetilde{\psi}}$ in Fourier space is equivalent to quasi-locality of $\mathcal{Q}_\psi$ in real space.

3. Proof of the main result

Let $f \in L^2(\mathbb{R})$ and recall the tail distributions $\alpha(s) = (\int_{|x|>s} |f(x)|^2 \, dx)^{1/2}$ and $\beta(s) = (\int_{|k|>s} |\widehat{f}(k)|^2 \, dk)^{1/2}$. Our main tool for proving the decay estimates for dispersion managed solitons is the following self-consistency bound on the tail distribution. For two functions g and h we write $g \lesssim h$ if there exists a constant $C > 0$ such that $g \leq Ch$.

Lemma 3.1 (Self-consistency estimate)**.** *Let $\omega > 0$ and $f \in L^2(\mathbb{R})$ be a weak solution of $\omega f = Q(f, f, f)$. Denote by α, respectively β, the tail distributions of f, respectively its Fourier transform. Then for all $s > 0$*

$$\alpha(3s) \lesssim (\alpha(s))^3 + \frac{\alpha(0)^2 \alpha(s)}{\sqrt{s}} \tag{3.1}$$

and

$$\beta(3s) \lesssim (\beta(s))^3 + \frac{\beta(0)^2\beta(s)}{\sqrt{s}}. \tag{3.2}$$

The implicit constant in the above estimates is bounded by CP_1/ω for some absolute constant C.

Remarks 3.2. (i) One can improve on this a little bit by replacing $\sqrt{s}$ with $\max(\sqrt{s}, 1)$ and one of the factors $\alpha(0)$, respectively $\beta(0)$, by $\alpha(0) - \alpha(s)$, respectively $\beta(0) - \beta(s)$ in the above bounds. As the proof of Corollary 3.3 shows, however, the precise value of the constant in the self-consistency bounds is not relevant for the decay estimates.

(ii) With (2.4) for the ground–state soliton and using (2.32) and (2.33) to chase the constants in the proof of Lemma 3.1 one sees that the rather explicit bounds

$$\overline{\alpha}(3s) \leq (\overline{\alpha}(s))^3 + 3\min(1, \frac{1}{\sqrt{s}})(1 - \overline{\alpha}(s))\overline{\alpha}(s) \tag{3.3}$$

and

$$\overline{\beta}(3s) \leq (\overline{\beta}(s))^3 + 3\min(1, \frac{0.78}{\sqrt{s}})(1 - \overline{\beta}(s))\overline{\beta}(s) \tag{3.4}$$

for the normalized tail distributions $\overline{\alpha}(s) = \alpha(s)/\alpha(0)$, respectively $\overline{\beta}(s) = \beta(s)/\beta(0)$, of the ground–state soliton hold. In the limit $s \to 0$, these bounds cannot be improved.

(iii) The self–consistency bounds for α and β provided by Lemma 3.1 are instrumental for our proof that α and β decay faster than any polynomial at infinity. The key property for this, as expressed by the the bounds (3.1) and (3.2), is the somewhat surprising fact that, despite the dispersion management equation being a highly non-local equation, the values of any weak solution of $f = Q(f, f, f)$ on the set $\{|x| > 3s\}$ can be controlled solely by the values of f on the slightly enlarged set $\{|x| > s\}$. This important property is due to the quasi-locality of $\mathcal{Q}$, as expressed in Lemma 2.10.

(iv) Although the self-consistency bounds (3.1) and (3.2) are not strong enough to yield exponential decay of α and β, they are not too far from the truth: A bound of the form

$$\alpha(3s) \lesssim (\alpha(s))^3 \quad \text{and } \beta(3s) \lesssim (\beta(s))^3, \tag{3.5}$$

i.e., dropping the second term, together with some decay of α can be bootstrapped to yield exponential decay of both α and β, see Remark 3.5.ii.

Proof of the self-consistency bounds. First we prove (3.1). Fix $s > 0$. Recall that f is a weak solution of $f = Q(f, f, f)$ if and only if $\langle g, f\rangle = \mathcal{Q}(g, f, f, f)$ for all $g \in L^2(\mathbb{R})$. Since the left hand side of (3.1) is

$$\alpha(3s) = \sup_{\substack{\mathrm{supp}\,(g) \subset (-\infty, -3s) \cup (3s, \infty) \\ \|g\|=1}} |\langle g, f\rangle|, \tag{3.6}$$

it remains to estimate $\mathcal{Q}(g, f, f, f)$ uniformly in $g \in L^2(\mathbb{R})$ with $\mathrm{supp}\, g \subset (-\infty, -3s) \cup (3s, \infty)$ and $\|g\|_2 = 1$. Let $I_s = [-s, s]$. We split f into its low and high space parts,

according to I_s: $f_< = f_{<,s} = f\chi_{I_s}$ and $f_> = f_{>,s} = f(1-\chi_{I_s})$, where χ_{I_s} is the characteristic function of the interval I_s, and use the multi-linearity of $\mathcal{Q}$ to rewrite

$$\begin{aligned}\omega\langle g,f\rangle = \mathcal{Q}(g,f,f,f) &= \mathcal{Q}(g,f_<,f_<,f_<) + \mathcal{Q}(g,f_>,f_>,f_>) \\ &\quad + \mathcal{Q}(g,f,f_<,f_>) + \mathcal{Q}(g,f_>,f,f_<) + \mathcal{Q}(g,f_<,f_>,f) \\ &= \mathcal{Q}(g,f_>,f_>,f_>) + \mathcal{Q}(g,f,f_<,f_>) + \mathcal{Q}(g,f_>,f,f_<) + \mathcal{Q}(g,f_<,f_>,f) \quad (3.7)\end{aligned}$$

where the last equality follows from the quasi-locality, $\mathcal{Q}(g,f_<,f_<,f_<) = 0$ from Lemma 2.10, since the supports of g and $f_<$ do not match by the definition of $f_<$.

Using Lemma 2.1, the first term on the right hand side of (3.7) is bounded by

$$|\mathcal{Q}(g,f_>,f_>,f_>)| \lesssim \|g\|_2\|f_>\|_2^3 = (\alpha(s))^3.$$

It remains to bound the last three terms of (3.7). Since, by assumption, the supports of g and $f_<$ are separated by at least $2s$, we can use Corollary 2.8 to see

$$|\mathcal{Q}(g,f,f_<,f_>)| \lesssim \frac{1}{\sqrt{s}}\|f\|\|f_<\|\|f_>\| \le \frac{1}{\sqrt{s}}\|f\|^2\|f_>\| = \frac{1}{\sqrt{s}}\alpha(0)^2\alpha(s).$$

The bounds for the other two terms are the same.

To prove the bound (3.2), one notes for any $\hat{g}\in L^2(\mathbb{R})$, $\langle \hat{g},\hat{f}\rangle = \langle g,f\rangle = \mathcal{Q}(g,f,f,f)$. Since

$$\beta(3s) = \sup_{\substack{\text{supp}\,(\widehat{g})\subset(-\infty,-3s)\cup(3s,\infty)\\ \|g\|=1}} |\langle \widehat{g},\widehat{f}\rangle|.$$

a proof similar to the above one, splitting f into its low and high frequency parts, gives the bound (3.2) for β. ∎

A-priori we only know that α and β decay to zero as $s\to\infty$ for an arbitrary $f\in L^2(\mathbb{R})$. The self consistency bounds of Lemma (3.1) allow us to bootstrap this and get some explicit super-polynomial decay. To see how this might work, assume for the moment that $h:\mathbb{R}_+\to\mathbb{R}_+$ obeys the bound

$$h(s) \lesssim (h(s))^3$$

for all $s\in\mathbb{R}_+$. Then, of course, for all $s\in\mathbb{R}_+$ either $h(s)=0$ or $1\lesssim h(s)$. So if in addition one knows that h decays to zero at infinity, it must already have compact support. The following makes this intuition precise.

Corollary 3.3 (= strengthening of Proposition 1.3)**.** *Let $\omega>0$ and $f\in L^2(\mathbb{R})$ a weak solution of $\omega f = Q(f,f,f)$. Then there exist s_0, respectively $\widehat{s}_0$, such that*

$$\alpha(s) \le \alpha(s_0)3^{1/4}3^{-(\log_3(\frac{s}{3s_0}))^2/4},$$
$$\beta(s) \le \beta(\widehat{s}_0)3^{1/4}3^{-(\log_3(\frac{s}{3\widehat{s}_0}))^2/4},$$

for all $s\ge s_0$, respectively $s\ge\widehat{s}_0$.

Remarks 3.4. (i) The above bounds are only effective when $s\ge 9s_0$, respectively $s\ge 9\widehat{s}_0$. Since α and β are monotone decreasing, they are bounded by $\alpha(0)=\beta(0)=\|f\|$ for small s.

(ii) Using Remark 1.4.ii, we get the same point-wise decay estimate for f.

(iii) The value of s_0, which is the only quantity in the decay estimate affected by the value of the constant in the self-consistency bound from Lemma 3.1, is determined in (3.9) below.

Proof. We prove only the first bound, the proof for the second is identical. By (3.1), we know that

$$\alpha(3s) \leq C\Big(\alpha(s)^2 + \frac{\alpha(0)^2}{\sqrt{s}}\Big)\alpha(s) \tag{3.8}$$

for some constant C. Since $f \in L^2(\mathbb{R})$, α is monotonically decreasing with $\alpha(\infty) = \lim_{s\to\infty} \alpha(s) = 0$. Thus there exists $s_0 < \infty$ such that

$$C\Big(\alpha(s_0)^2 + \frac{\alpha(0)^2}{\sqrt{s_0}}\Big) \leq 3^{-1/4}. \tag{3.9}$$

The monotonicity of α together with (3.9) and (3.8) yield the a-priori bound

$$\alpha(3s) \leq 3^{-1/4}\alpha(s) \qquad \text{for all } s \geq s_0.$$

Putting $\gamma(t) := \log_3(\alpha(3^t))$ and $t_0 = \log_3(s_0)$, we see that

$$\gamma(t+1) \leq \gamma(t) - \frac{1}{4} \qquad \text{for all } t \geq t_0. \tag{3.10}$$

With $\widetilde{\gamma}_1(t) = \gamma(t) + t/4$ this is equivalent to

$$\widetilde{\gamma}_1(t+1) - \widetilde{\gamma}_1(t) = \gamma(t+1) - \gamma(t) + \frac{1}{4} \leq 0 \qquad \text{for all } t \geq t_0,$$

which shows that $\widetilde{\gamma}_1$ is sub-periodic for $t \geq t_0$. In particular,

$$\widetilde{\gamma}_1(t) \leq \sup_{t' \in [t_0, t_0+1)} \widetilde{\gamma}_1(t') \leq \gamma(t_0) + \frac{t_0+1}{4} = \log_3\big(\alpha(s_0)(3s_0)^{1/4}\big)$$

for all $t \geq t_0$ since $\alpha(s)$ and hence also $\gamma(t)$ is decreasing. In turn, this yields $\gamma(t) \leq \log_3\big(\alpha(s_0)(3s_0)^{1/4}\big) - \frac{t}{4}$, or, equivalently,

$$\alpha(s) \leq \alpha(s_0)\Big(\frac{3s_0}{s}\Big)^{1/4} \qquad \text{for all } s \geq s_0. \tag{3.11}$$

Now we bootstrap this once. Plugging (3.11) back into (3.8) and using (3.9) one gets

$$\begin{aligned}\alpha(3s) &\leq C\Big(\alpha(s_0)^2 + \frac{\alpha(0)^2}{\sqrt{3s_0}}\Big)\Big(\frac{3s_0}{s}\Big)^{1/2}\alpha(s) \\ &\leq 3^{-1/4}\Big(\frac{3s_0}{s}\Big)^{1/2}\alpha(s)\end{aligned}$$

for all $s \geq s_0$. Hence (3.10) is improved to

$$\gamma(t+1) \leq \gamma(t) - \frac{1}{4} + \frac{t_0+1-t}{2} \qquad \text{for all } t \geq t_0. \tag{3.12}$$

With $\widetilde{\gamma}_2(t) := \gamma(t) + (t - t_0 - 1)^2/4$ the bound (3.12) is equivalent to

$$\widetilde{\gamma}_2(t+1) - \widetilde{\gamma}_2(t) \leq 0 \qquad \text{for all } t \geq t_0.$$

Hence, for all $t \geq t_0$,

$$\widetilde{\gamma}_2(t) \leq \sup_{t\in[t_0,t_0+1]} \widetilde{\gamma}_2(s) \leq \gamma(t_0) + \frac{1}{4}.$$

Equivalently,

$$\gamma(t) \leq \gamma(t_0) + \frac{1}{4} - \frac{(t-t_0-1)^2}{4} \quad \text{for all } t \geq t_0,$$

which yields the claimed inequality for $\alpha(s)$. Given (3.2), the same proof applies to the tail distribution of $\widehat{f}$. ∎

Remarks 3.5. (i) There are non exponentially decaying functions which obey the self-consistency bounds of Lemma 3.1. For example,

$$g(s) = 3^{-(\log_3(\frac{s}{3s_0}))_+^2},$$

with $(r)_+ = \max(0, r)$, obeys the bound

$$g(3s) \leq \sqrt{\frac{3s_0}{s}}\, g(s)$$

for all $s > 0$. Thus the bounds given in Lemma 3.1 are not strong enough to yield the conjectured exponential decay for the dispersion managed soliton.

(ii) The self-consistency bounds given by Lemma 3.1 are not too far from the truth. A bound of the form

$$\alpha(3s) \lesssim \alpha(s)^3 \tag{3.13}$$

is not only consistent with exponential decay of α, but, together with the a-priori decay $\lim_{s\to\infty} \alpha(s) = 0$, implies exponential decay of α. To see this, let us assume first

$$\alpha(3s) \leq \alpha(s)^3$$

for all $s \geq 0$. With $\gamma(t) = \log_3 \alpha(3^t)$, this is equivalent to

$$\gamma(t+1) \leq 3\gamma(t)$$

for all t and iterating this bound yields

$$\gamma(t) \leq 3^n \gamma(t-n) \quad \text{for all } t \text{ and all } n \in \mathbb{N}_0. \tag{3.14}$$

Since $\gamma(t) \to -\infty$, as $t \to \infty$, we can choose t_0 such that

$$3\mu := -\gamma(t_0) > 0.$$

With this choice (3.14) implies

$$\gamma(t_0 + n) \leq -\mu 3^{n+1}$$

for all n. Since α and hence γ is decreasing, this gives

$$\gamma(t) \leq -\mu 3^{n+1} \quad \text{for all } t \in [t_0+n, t_0+n+1]$$

or, equivalently,

$$\alpha(s) \leq 3^{-\mu 3^{n+1}} \quad \text{for all } s \in [s_0 3^n, s_0 3^{n+1}]$$

where $s_0 = 3^{t_0}$. Thus

$$\alpha(s) \leq 3^{-\mu s/s_0} \quad \text{for all } s \geq s_0.$$

If $\alpha(3s) \le C\alpha(s)^3$ for all $s \ge 0$ with $C > 0$, then, with $\widetilde{\alpha}(s) = \sqrt{C}\alpha(s)$,

$$\widetilde{\alpha}(3s) \le \widetilde{\alpha}(s)^3 \quad \text{for all } s \ge 0.$$

By the above argument $\widetilde{\alpha}$, hence also α, decays exponentially if a bound of the form (3.13) holds.

For the last two results, which finish the proof of Proposition 1.5 and Theorem 1.1, it is convenient to introduce one more notation: for $x \in \mathbb{R}$ let $\langle x \rangle = \sqrt{1+x^2}$.

Corollary 3.6 (= Proposition 1.5)**.** *If $f \in L^2(\mathbb{R})$ is a weak solution of $\omega f = Q(f,f,f)$ with $\omega > 0$, then the functions $x \mapsto \langle x \rangle^n f(x)$ and $k \mapsto \langle k \rangle^m \widehat{f}(k)$ are both square integrable for all $m, n \in \mathbb{N}$. In particular, both f and its Fourier transform are C^∞ functions with all their derivatives, of arbitrary order, square integrable functions.*

Proof. From Corollary 3.3 we know $\beta(s)^2 = \int_{|k|>s} |\widehat{f}(k)|^2\, dk$ decays faster than any polynomial. Thus

$$\begin{aligned}\int_{\mathbb{R}} \langle k \rangle^m |\hat{f}(k)|^2\, dk &= -\int_0^\infty \langle s \rangle^m d(\beta(s)^2) \\ &= \int_0^\infty m\langle s \rangle^{m-2} s(\beta(s))^2\, ds + \beta(0)^2 < \infty,\end{aligned}$$

the integration by parts is justified due to the super-polynomial decay of β. The argument for $x \mapsto \langle x \rangle^n f(x)$ is identical. Thus both f and $\widehat{f}$ are in all the Sobolev spaces $H^s(\mathbb{R})$ for arbitrary $s > 0$ and the smoothness of f and $\widehat{f}$ follows from the Sobolev embedding theorem. ∎

These two corollaries together with the following lemma finish the proof of our main Theorem 1.1.

Lemma 3.7. *A function $f : \mathbb{R} \to \mathbb{C}$ is a Schwartz function if and only if $x \mapsto \langle x \rangle^n f(x)$ is square integrable for all $n \in \mathbb{N}$ and all weak derivatives of f are square integrable.*

Proof. Let $D = -i\partial_x$. Lemma 1 on page 141 in [30] tells us that f is a Schwartz function, that is,

$$\|f\|_{n,m,\infty} = \sup_{x\in\mathbb{R}} |\langle x \rangle^n D^m f(x)| < \infty$$

for all $n, m \in \mathbb{N}_0$, if and only if

$$\|f\|_{n,m,2} = \Big(\int |\langle x \rangle^n D^m f(x)|^2\, dx \Big)^{1/2} < \infty$$

for all $n, m \in \mathbb{N}_0$. In particular, for any Schwartz function f, $\|f\|_{n,0,2}$ and $\|f\|_{0,m,2}$ are finite for all $m, n \in \mathbb{N}_0$, that is, the functions $x \mapsto \langle x \rangle^n f(x)$ and $x \mapsto D^m f(x)$ are both square integrable for any $n, m \in \mathbb{N}_0$.

To prove the converse, it is clearly enough to show that for all $m, n \in \mathbb{N}$ finiteness of $\|f\|_{2n,0,2}$ and $\|f\|_{0,m+j,2}$ for $j = 0, 1, \ldots, m$ imply $\|f\|_{n,m,2} < \infty$. For any $\epsilon > 0$ define $\langle x \rangle_\varepsilon = \langle x \rangle / \langle \varepsilon x \rangle$. By Lemma 3.8 below all derivatives of $\langle x \rangle_\varepsilon^{2n}$ are bounded for

$0 < \varepsilon \le 1$ and all $n \in \mathbb{N}_0$. In particular, for any $0 < \varepsilon \le 1$ and all $n, m \in \mathbb{N}$ we also have $\langle x\rangle_\varepsilon^{2n} g \in H^m(\mathbb{R})$ as soon as $g \in H^m(\mathbb{R})$.

Now let n, $m \in \mathbb{N}_0$ and f such that $\|f\|_{2n,0,2} < \infty$ and $\|f\|_{0,m+j,2} < \infty$ for $j = 0, 1, \dots, m$. In particular, $f \in H^{2m}(\mathbb{R})$. In the following, it is convenient to think of D^m as a self-adjoint operator with domain $H^m(\mathbb{R})$. By the Leibnitz rule for derivatives,

$$\begin{aligned}
&\int |\langle x\rangle_\varepsilon^n D^m f|^2 dx = \langle \langle x\rangle_\varepsilon^n D^m f, \langle x\rangle_\varepsilon^n D^m f\rangle = \langle f, D^m \langle x\rangle_\varepsilon^{2n} D^m f\rangle \\
&= \langle f, \sum_{j=0}^m \binom{m}{j} D^{m-j}\langle x\rangle_\varepsilon^{2n} D^{j+m} f\rangle = \sum_{j=0}^m \binom{m}{j} \langle f D^{m-j}\langle x\rangle_\varepsilon^{2n}, D^{j+m} f\rangle \\
&\le \sum_{j=0}^m \binom{m}{j} \|f D^{m-j}\langle x\rangle_\varepsilon^{2n}\| \|D^{j+m} f\| \le \sum_{j=0}^m \binom{m}{j} \|f \langle x\rangle^{(2n-m+j)_+}\| \|D^{j+m} f\| \\
&\le \sum_{j=0}^m \binom{m}{j} \|f\|_{2n,0,2} \|f\|_{0,j+m,2},
\end{aligned}$$

where the last inequality uses Lemma 3.8 and $\langle x\rangle^{(2n-m+j)_+} \le \langle x\rangle^{2n}$ for all $j = 0, 1, \dots, m$. Thus, by monotone convergence,

$$\int |\langle x\rangle^n D^m f|^2 dx = \lim_{\varepsilon\to 0} \int |\langle x\rangle_\varepsilon^n D^m f|^2 dx \le \sum_{j=0}^m \binom{m}{j} \|f\|_{2n,0,2}\|f\|_{0,j+m,2} < \infty$$

by the assumptions on f. Hence $\|f\|_{n,m,2} < \infty$. ∎

To finish the proof of Lemma 3.7, we need the

Lemma 3.8. *For $\varepsilon \ge 0$ let $\langle x\rangle_\varepsilon = \frac{\langle x\rangle}{\langle \varepsilon x\rangle}$. Then, with $D = -i\partial_x$, one has for all $\eta \in \mathbb{R}$ and all $m \in \mathbb{N}_0$*

$$|D^m(\langle x\rangle_\varepsilon^\eta)| \lesssim \langle x\rangle^{(\eta-m)_+} = \begin{cases} \langle x\rangle^{\eta-m} & \text{for} \quad m \le \eta \\ 1 & \text{for} \quad m > \eta \end{cases} \tag{3.15}$$

uniformly in $\varepsilon \in [0,1]$ with the implicit constant depending only on η and m.

Proof. A straightforward induction on m shows that for $j = 0, 1, \dots, m$ there are polynomials $p_j = p_{j,m,\eta}$ of degree at most j such that for all $x \in \mathbb{R}$

$$D^m(\langle x\rangle^\eta) = \sum_{j=0}^m p_j(x)\langle x\rangle^{\eta-m-j}.$$

This immediately implies the bound

$$|D^m(\langle x\rangle^\eta)| \lesssim \langle x\rangle^{\eta-m} \tag{3.16}$$

for all $\eta \in \mathbb{R}$ and all $m \in \mathbb{N}_0$ with a constant depending only on m and η. The Leibnitz rule, the triangle inequality, (3.16), and the Binomial formula imply

$$|D^m(\langle x\rangle_\varepsilon^\eta)| = |D^m(\langle x\rangle^\eta \langle \varepsilon x\rangle^{-\eta})| \le \sum_{j=0}^m \binom{m}{j} |D^{m-j}\langle x\rangle^\eta||D^j\langle \varepsilon x\rangle^{-\eta}|$$

$$\lesssim \sum_{j=0}^{m} \binom{m}{j} \langle x\rangle^{\eta-(m-j)} \langle \varepsilon x\rangle^{-\eta-j} \varepsilon^j = \frac{\langle x\rangle^{\eta}}{\langle \varepsilon x\rangle^{\eta}} \sum_{j=0}^{m} \binom{m}{j} \langle x\rangle^{-(m-j)} \langle \varepsilon x\rangle^{-j} \varepsilon^j$$
$$= \frac{\langle x\rangle^{\eta}}{\langle \varepsilon x\rangle^{\eta}} \Big(\frac{1}{\langle x\rangle} + \frac{\varepsilon}{\langle \varepsilon x\rangle} \Big)^m \lesssim \frac{\langle x\rangle^{\eta-m}}{\langle \varepsilon x\rangle^{\eta}} = \langle x\rangle_\varepsilon^{\eta-m} \langle \varepsilon x\rangle^{-m} \le \langle x\rangle_\varepsilon^{\eta-m}$$

since $\varepsilon\langle x\rangle \le \langle \varepsilon x\rangle$ for all x and all $0 \le \varepsilon \le 1$. This proves (3.15) since $1 \le \langle \varepsilon x\rangle \le \langle x\rangle$ for all x, and $0 \le \varepsilon \le 1$. ∎

Remark 3.9. Using the multidimensional Binomial and Leibnitz formulas, see Theorem 1.2 in [29], the corresponding statement of Lemma 3.7 and Lemma 3.8 hold also on $\mathbb{R}^d$ with virtually identical proofs.

Acknowledgment: It is a pleasure to thank Vadim Zharnitsky for instructive discussions on the dispersion management technique and introducing us to the problem of decay estimates for dispersion management solitons. We would also like to Tony Carbery, Maria and Thomas Hoffmann-Ostenhof, and Rick Laugesen for discussions. Young-Ran Lee thanks the School of Mathematics of the University of Birmingham, UK, for their warm hospitality.

References

[1] M.J. Ablowitz and G. Biondini, *Multiscale pulse dynamics in communication systems with strong dispersion management.* Opt. Lett. **23** (1998), 1668–1670.

[2] G. P. Agrawal, *Nonlinear Fiber Optics.* Second Edition (Optics and Photonics), Academic Press, San Diego, 1995.

[3] J. Bourgain, *Fourier transform restriction phenomena for lattice subsets and applications to nonlinear evolution equations I. Schrödinger equation.* Geom. Funct. Anal. **3** (1993), 107–156.

[4] J. Bourgain, *Fourier transform restriction phenomena for lattice subsets and applications to nonlinear evolution equations II. The KdV equation.* Geom. Funct. Anal. **3** (1993), 209–262.

[5] Thierry Cazenave, *Semilinear Schrödinger equations,* Courant Lecture Notes in Mathematics **10**, AMS, Providence, Rhode Island, 2003.

[6] A.R. Chraplyvy, A.H. Gnauck, R.W. Tkach, and R.M. Derosier, *8 10 Gb/s transmission through 280 km of dispersion-managed fiber.* IEEE Phot. Tech. Lett. **5** (1993), 1233–1235.

[7] L. G. Cohen, C. Lin, and W. G. French, *Tailoring zero chromatic dispersion into the 1.5–1.6 μm low-loss spectral region of single-mode fibres.* Electron. Lett. **15** (1979), 334–335.

[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, *A refined global well-posedness result for Schrödinger equations with derivative.* SIAM J. Math. Anal. **34** (2002), 64–86.

[9] Damiano Foschi, *Maximizers for the Strichartz inequality.* J. Eur. Math. Soc. **9** (2007), 739–774.

[10] I. Gabitov and S.K. Turitsyn, *Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation.* Opt. Lett. **21** (1996), 327–329.

[11] I. Gabitov and S.K. Turitsyn, *Breathing solitons in optical fiber links.* JETP Lett. **63** (1996) 861.

[12] Jean Ginibre and Giorgio Velo, *The global Cauchy problem for the nonlinear Schrödinger equation.* Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 3009–327.

[13] Dirk Hundertmark and Young-Ran Lee, *On the existence of dispersion managed solitons for vanishing residual dispersion.* In preparation.

[14] Dirk Hundertmark and Vadim Zharnitsky, *On sharp Strichartz inequalities for low dimensions.* International Mathematics Research Notices, vol. 2006, Article ID 34080, 18 pages, 2006. doi:10.1155/IMRN/2006/34080

[15] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations.* Indiana Univ. Math. J. **40** (1991), 33–68.

[16] Markus Kunze, *On a variational problem with lack of compactness related to the Strichartz inequality.* Calc. Var. Partial Differential Equations **19** (2004), no. 3, 307–336.

[17] S. Kumar and A. Hasegawa, *Quasi-soliton propagation in dispersion-managed optical fibers.* Opt. Lett. **22** (1997), 372–374.

[18] C. Kurtzke, *Suppression of fiber nonlinearities by appropriate dispersion management.* IEEE Phot. Tech. Lett. **5** (1993), 1250–1253.

[19] T. Lakoba and D.J. Kaup, *Shape of the stationary pulse in the strong dispersion management regime.* Electron. Lett. **34** (1998), 1124–1125.

[20] L.D. Landau and E.M. Lifshitz, *Course of theoretical physics. Vol. 1. Mechanics.* Third edition. Pergamon Press, Oxford-New York-Toronto, Ont., 1976.

[21] C. Lin, H. Kogelnik, and L. G. Cohen, *Optical pulse equalization and low dispersion transmission in singlemode fibers in the 1.3–1.7 μm spectral region.* Opt. Lett. **5** (1980), 476-478.

[22] P. L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, part 1 and 2.* Annales de l'institut Henri Poincaré (C) Analyse non linéaire **1** no. 2 and no. 4 (1984), 109–145 and 223–283.

[23] Pavel M. Lushnikov, *Dispersion-managed soliton in a strong dispersion map limit.* Optics Letters **26** (October 15, 2001), no. 20, 1535–1537.

[24] Pavel M. Lushnikov, *Oscillating tails of dispersion-managed soliton.* J. Opt. Soc. Am. B **21** (2004), 1913–1918.

[25] P.V. Mamyshev and N.A. Mamysheva, *Pulseoverlapped dispersion-managed data transmission and intrachannel four-wave mixing.* Opt. Lett. **24** (1999) 1454–1456.

[26] L.F. Mollenauer, A. Grant, X. Liu, X. Wei, C. Xie, and I. Kang, *Experimental test of dense wavelengthdivision multiplexing using novel, periodic-group-delaycomplemented dispersion compensation and dispersionmanaged solitons.* Opt. Lett. **28** (2003), 2043–2045.

[27] L.F. Mollenauer, P.V. Mamyshev, J. Gripp, M.J. Neubelt, N. Mamysheva, L. Grüner-Nielsen and T. Veng, *Demonstration of massive wavelength-division multiplexing over transoceanic distances by use of dispersionmanaged solitons.* Opt. Lett. **25** (1999), 704–706.

[28] T. Ozawa and Y. Tsutsumi, *Space-time estimates for null gauge forms and nonlinear Schrödinger equations.* Differential Integral Equations **11** (1998), 201–222.

[29] X. S. Raymond, *Elementary introduction to the theory of speudodifferential operators.* Studies in Advanced Mathematics. CRC Press, Boca Raton 1991.

[30] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis.* Revised and Enlarged Edition New York: Academic Press 1980.

[31] T. Schäfer, E. W. Laedke, M. Gunkel, C. Karle, A. Posth, K. H. Spatschek, and S. K. Turitsyn, *Optimization of dispersion-managed optical fiber lines.* IEEE J. Light. Tech. **20**(2002), 946–952.

[32] Milena Stanislavova, *Regularity of ground state solutions of DMNLS equations.* J. Diff. Eq. **210** (2005), no. 1, 87–105.

[33] R. S. Strichartz, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations,* Duke math. J. **44** (1977), 705–714.

[34] Cathrine Sulem and Pierre-Louis Sulem, *The non-linear Schrödinger equation. Self-focusing and wave collapse.* Applied Mathematical Sciences, **139**. Springer-Verlag, New York, 1999.

[35] T. Tao, *Multilinear weighted convolution of L2-functions, and applications to nonlinear dispersive equations.* Amer. J. Math. **123** (2001), no. 5, 839–908.

[36] H. Toda, K. Hamada, Y. Furukawa, Y. Kodama, and S. Seikai, *Experimental Evaluation of Gordon-Haus timing jitter of dispersion managed solitons.* In Proceedings of the European Conference on Optical Communication (ECOC99), Vol. 1 (1999), pp. 406–407.

[37] S.K. Turitsyn, N.J. Doran, J.H.B. Nijhof, V.K. Mezentsev, T. Schäfer, and W. Forysiak, in *Optical Solitons: Theoretical challenges and industrial perspectives,* V.E. Zakharov and S. Wabnitz, eds. (Springer Verlag, Berlin, 1999), p. 91.

[38] S. K. Turitsyn, E. G. Shapiro, S. B. Medvedev, M. P. Fedoruk, and V. K. Mezentsev, *Physics and mathematics of dispersion-managed optical solitons,* Comptes Rendus Physique, Académie des sciences/Éditions scientifiques et médicales **4** (2003), 145–161.

[39] V. Zharnitsky, E. Grenier, C.K.R.T. Jones, and S.K. Turitsyn, *Stabilzing effects of dispesion management,* Physica D **152-153** (2001), 794–817.

School of Mathematics, Watson Building, University of Birmingham, Edgbaston, Birmingham, B15 2TT, UK, on leave from Department of Mathematics, Altgeld Hall, University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801.

E-mail address: dirk@math.uiuc.edu

Department of Mathematical Sciences KAIST (Korean Advanced Institute of Science and Technology), 335 Gwahangno, Yuseong-gu, Daejeon, 305-701, Republic of Korea.

E-mail address: youngranlee@kaist.ac.kr